THE GREEN'S TENSOR FOR AN ELASTIC ISOTROPIC MEDIUM WITH SPATIAL DISPERSION

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A model of an elastic medium of simple structure with spatial dispersion was examined in [1, 2]. In the isotropic case the equations for the displacements in the (k, ω)-representation formally coincide with the equations of the ordinary theory of elasticity, but the Lamé constants λ, μ depend on the modulus of the wave vector $\mathbf{k}$ and (in the case of complex structure) on the frequency ω. Here, $|\mathbf{k}| \leq \varkappa = \pi/a$, where $\varkappa$ is the Debye radius, and a a characteristic scale parameter of the medium. The purpose of this note is to construct for the given model an explicit expression for the static Green's tensor $G_{\alpha\beta}$ in the $\mathbf{r}$-representation.

The starting equation is the known expression for $G_{\alpha\beta}$ in the k-representation [1]

$$G_{\alpha\beta}(\mathbf{k}) = \frac{1}{k^2\mu(k)}\left[\delta_{\alpha\beta} - \frac{\lambda(k)+\mu(k)}{\lambda(k)+2\mu(k)}\frac{k_\alpha k_\beta}{k^2}\right]\left(\begin{matrix} k = |\mathbf{k}| \\ r = |\mathbf{r}| \end{matrix}\right). \quad (1)$$

Here and in what follows $f(\mathbf{k})$ denotes the Fourier transform of $f(\mathbf{r})$.

For our purpose it is convenient to represent $G_{\alpha\beta}$ in the form of a sum of the longitudinal (l) and transverse (t) Green's tensors

$$G_{\alpha\beta}{}^{l}(\mathbf{k}) = \frac{k_\alpha k_\beta}{\rho k^2 \omega_l{}^2(k)}, \quad G_{\alpha\beta}{}^{t}(\mathbf{k}) = \frac{1}{\rho\omega_t{}^2(k)}\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right), \quad (2)$$

$$\omega_l{}^2(k) = \frac{k^2[\lambda(k)+2\mu(k)]}{\rho}, \qquad \omega_t{}^2(k) = \frac{k^2\mu(k)}{\rho}. \quad (3)$$

Here ρ is the density. For a medium of simple structure the equations $\omega = \omega_i(k)$ ($i = l$, t) determine the longitudinal and transverse branches of the vibrations while the functions $\omega_i(k)$ must satisfy the relations

$$d\omega_i(0)/dk = c_i, \qquad d\omega(\varkappa)/dk = 0, \quad (4)$$

where c_i are the speeds of sound at $k = 0$. We write the expressions $\omega_i(k)$ in the form

$$\omega_i{}^2 = c_i{}^2 k^2 / g_i(k^2), \quad (5)$$

where $g_i(x)$ is a suitable approximating polynomial satisfying conditions (4). Obviously, the case $g_i = 1$ corresponds to the Debye model [3]. A more realistic model is obtained by setting

$$g_i(k^2) = 1 + \gamma_i(k/\varkappa)^2 + (k/\varkappa)^4 \qquad (-2 < \gamma_i < \infty). \quad (6)$$

The corresponding family of dispersion curves depending on the parameter γ_i is shown in the figure. The straight line corresponds to the Debye model, $\omega_i{}^0 = c_i\varkappa$ is the Debye frequency. The dashed curve corresponds to the Born-Karman model [3] and, as may be seen from the figure, almost coincides with the curve $\gamma_i = 0.5$. In the more general case the parameter γ_i is found from the relation

$$\omega_i(\varkappa)/\omega_i{}^0 = 1/\sqrt{2+\gamma_i}. \quad (7)$$

In the r-representation all the admissible functions (including the δ-function and the Green's tensor) belong to the class of entire analytic functions of degree $\leq \varkappa$. This is a consequence of the Fourier spectra of the functions being cut off by the Debye radius or, more graphically, the existence of an elementary unit of length in r-space [1].

For functions depending only on r the Fourier inversion formula has the form

$$f(r) = \frac{1}{2\pi^2 r}\int_0^\varkappa k \sin kr f(k)\, dk. \quad (8)$$

From this, in particular, there follows the expression for the three-dimensional δ function in the indicated class of functions

$$\delta_\varkappa(\mathbf{r}) = \frac{\varkappa}{2\pi^2 r^2}\left(\frac{\sin\varkappa r}{\varkappa r} - \cos\varkappa r\right). \quad (9)$$

We note that $\delta_\varkappa(r)$ goes over into the ordinary δ functions as $\varkappa \to \infty$. For the Debye model $\omega_i(k) = c_i k$ ($i = l$, t), and from (2), using (8), we obtain an explicit expression for the Green's tensors

$$G_{\alpha\beta}{}^{l}(\mathbf{r}) = \frac{1}{\rho c_l{}^2}\partial_\alpha\partial_\beta F(r),$$

$$G_{\alpha\beta}{}^{t}(\mathbf{r}) = \frac{1}{\rho c_t{}^2}[\Delta F(r)\delta_{\alpha\beta} - \partial_\alpha\partial_\beta F(r)], \quad (10)$$

$$F(r) = \frac{1}{4\pi^2\varkappa}\left(\varkappa r \operatorname{Si}\varkappa r + \frac{\sin\varkappa r}{\varkappa r} + \cos\varkappa r\right). \quad (11)$$

Here, Si x is the integral sine. In the general case, substituting (5) into (2), we find

$$G_{\alpha\beta}{}^{l}(\mathbf{r}) = \rho^{-1}c_l{}^{-2}\ \partial_\alpha\partial_\beta F_l(r),$$

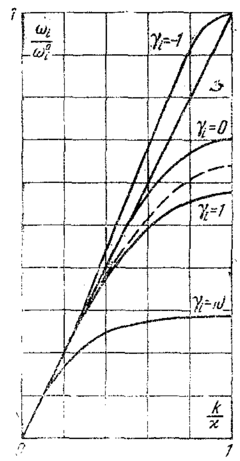
$$G_{\alpha\beta}{}^{t}(\mathbf{r}) = \rho^{-1}c_t{}^{-2}[\Delta F_t(r)\delta_{\alpha\beta} - \partial_\alpha\partial_\beta F_t(r)], \quad (12)$$

$$F_i(r) = g_i(-\Delta)F(r) \qquad (i = l, t). \quad (13)$$

In particular, for model (6)

$$F_i(r) = F(r) - {}^1/_2\,\gamma_i\pi^{-2}\varkappa^{-2}r^{-1}\operatorname{Si}\varkappa r - \varkappa^{-4}\delta_\varkappa(r). \quad (14)$$

When $r \gg a$ or, what amounts to the same thing, as $\varkappa \to \infty$ expressions (10) and (12) go over into the known equations of the ordinary theory of elasticity. However, as pointed out above, as distinct from the ordinary theory of elasticity, the Green's tensor constructed does not have a singularity at zero. This makes it possible, for example, to calculate the elastic energy W for a concentrated force Q:



$$W = \frac{1}{2}Q^\alpha Q^\beta G_{\alpha\beta}(0) = \frac{Q^2\varkappa}{10\pi^2\rho}\left(\frac{\Gamma_l}{c_l{}^2} + \frac{2\Gamma_t}{c_t{}^2}\right). \quad (15)$$

For model (6) $\Gamma_i = 1 + {}^5/_{18}\,\gamma_i$ ($i = l$, t), while for the Born-Karman model we can set $\gamma_i = 0.5$.

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